Initially Stressed Mindlin Plates

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Equations of motion for a transversely isotropic plate in a general state of nonuniform initial stress where the effects of transverse shear and rotary inertia are included are derived by two methods. The first method is to perturb the nonlinear equations of elasticity by an incremental deformation. The resulting equations are linearized and integrated through the thickness of the plate to obtain the plate equations. The second method is to derive nonlinear equations of motion for a thick plate variationally by Hamilton's principle. These equations are then perturbed and suitably linearized to obtain the same equations as were obtained by the first method. A reduced set of equations for a thin plate are also given. Finally, the thick plate equations are solved for a simply supported rectangular plate in a state of uniform compressive stress plus a uniform bending stress both acting in the same direction.

I. Introduction

TWO methods are employed to derive the governing equations of a transversely isotropic plate in a general state of nonuniform initial stress where the effects of transverse shear and rotary inertia are included. The first method uses the nonlinear equations of elasticity given in terms of Trefftz stress components and material coordinates.[1,2] These equations are perturbed by an incremental deformation.[3] The resulting equations are subsequently linearized and simplified by dropping products of incremental stresses with incremental and initial displacement gradients. Following the procedure used by Mindlin[4] when he derived the equations for a thick plate, the linearized perturbed equations are integrated through the thickness of the plate. The incremental deformation is assumed to obey Hooke's law so that by assuming a displacement field that accounts for transverse shearing[4] the initially stressed plate displacement equations can be found.

Because the equations were found to differ in some respects from a previous formulation[5] in addition to giving seemingly clearer expressions for the boundary and lateral loading terms a variational derivation of the equations was also performed. The variational procedure used is based on Hamilton's principle and is similar to that used in Ref. 5 except that no terms are dropped unnecessarily and the Trefftz definition of stress is used throughout rather than a mixture of definitions. The procedure consists of deriving nonlinear equations for a thick plate that are subsequently perturbed and linearized. The resulting equations are the same as those obtained by the first method.

As an aside, the equations for a thin plate in a state of initial stress are obtained by simplifying the equations for an initially stressed thick plate in an appropriate fashion.

The equations for an initially stressed Timoshenko beam have been derived by Brunelle[6] who used the method of integrating the plane stress equations for a perturbed elastic body through the thickness of the beam. They agree with the equations given in this paper when the present equations are one-dimensionalized to conform to the beam problem.

The equations for the initially stressed thick plate were applied to a simply supported rectangular plate subjected to uniform in-plane compression (or tension) and a uniform bending stress acting in the x_1-direction. The characteristic equation for determining natural frequencies and buckling loads is given. Varying the parameters in the characteristic equation three stability problems were considered, a thin plate subjected to uniform compressive stress, a thick plate subjected to uniform compressive stress and a thick plate subjected to uniform compressive and bending stresses.

The equations derived here can be used to study nonconservative as well as conservative stability and dynamic problems for sundry states of initial stress. In Ref. 6 results are given for a number of conservative and nonconservative buckling and vibration problems for an initially stressed Timoshenko beam that are useful in understanding the equations for an initially stressed thick plate.

II. Nonlinear Equations

The equations of equilibrium in component form can be expressed in terms of Trefftz stress components as[1]

$$\frac{\partial}{\partial x_i}\left[(\delta_{js}+\partial u_s/\partial x_j)\sigma_{ij}^*\right]+X_s^*=0 \tag{1}$$

where the x_i are material (Lagrangian) coordinates that originally coincided with a spatial (fixed) Cartesian coordinate system. The u_s are Cartesian components of the displacement vector given with respect to the spatial coordinate system. The Trefftz stresses σ_{ij}^* are components of the stress vector taken with respect to undeformed areas and resolved along material coordinates using the lattice vectors ($\partial \mathbf{r}/\partial x_i$, $\mathbf{r}$ being the position vector) rather than unit vectors. The Trefftz stresses are symmetric,

$$\sigma_{ij}^*=\sigma_{ji}^* \tag{2}$$

The X_s^* are the body force components referred to the spatial frame per unit of initial volume. δ_{js} is the Kronecker delta

$$\delta_{js}=\begin{cases}0 & j\neq s\\ 1 & j=s\end{cases} \tag{3}$$

A repeated index implies summation over the range of the index. For notational convenience a comma preceding an index implying partial differentiation with respect to the appropriate material coordinate will be used in the sequel.

The boundary traction conditions are

$$p_s^*=\sigma_{ij}^*n_i(\delta_{js}+u_{s,j}) \tag{4}$$

The n_i are the components of the unit normal given with respect to the spatial frame. The p_s^* are components of the applied

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surface traction referred to the spatial frame and taken with respect to initial areas. A brief derivation of these component forms of the equations from the vector form is given in the Appendix.

If relative extensions and shears are small then the final areas and volumes are equal to the initial areas and volumes so that $\sigma_{ij}^* = \sigma_{ij}$, $X_s^* = X_s$, and $p_s^* = p_s$ where σ_{ij}, X_s, and p_s are the actual stresses, body forces and surface tractions, respectively. These approximations will be adopted in the work that follows. Thus, the governing equations become

$$[(\delta_{js}+u_{s,j})\sigma_{ij}]_{,i}+X_s = 0 \qquad (5)$$

$$p_s = \sigma_{ij} n_i(\delta_{js}+u_{s,j}) \qquad (6)$$

III. Perturbed Equations

The problem of interest here is that of a body that is in a state of nonuniform initial stress which is in static equilibrium and subjected to a time varying incremental deformation. Following a technique described by Bolotin[3] the following quantities are introduced

$$\hat{u}_s = u_s+\bar{u}_s \qquad (7)$$

$$\hat{\sigma}_{ij} = \sigma_{ij}+\bar{\sigma}_{ij} \qquad (8)$$

$$\hat{p}_s = p_s+\Delta p_s+\bar{p}_s \qquad (9)$$

$$\hat{X}_s = X_s+\Delta X_s+\bar{X}_s-\rho\ddot{\bar{u}}_s \qquad (10)$$

where, for example, $\hat{u}_s$, u_s, and $\bar{u}_s$ represent the final displacement, initial displacement, and the perturbing displacement, respectively. The terms Δp_s and ΔX_s represent changes in the initial surface traction and body force due to the perturbation. The term $-\rho\ddot{\bar{u}}_s$ is the inertia force due to the perturbation and the superior double dot denotes the second partial derivative with respect to time.

The final state of stress satisfies Eqs. (5) and (6)

$$[(\delta_{js}+\hat{u}_{s,j})\hat{\sigma}_{ij}]_{,i}+\hat{X}_s = 0 \qquad (11)$$

$$\hat{p}_s = \hat{\sigma}_{ij} n_i(\delta_{js}+\hat{u}_{s,j}) \qquad (12)$$

Substitution of Eqs. (7–10) into Eqs. (11) and (12) and using the fact that the initial displacements and stresses satisfy Eqs. (5) and (6) too gives the following equation for the incremental stresses and displacements:

$$(\sigma_{ij}\bar{u}_{s,j})_{,i}+[\bar{\sigma}_{ij}(\delta_{sj}+u_{s,j}+\bar{u}_{s,j})]_{,i}+\Delta X_s+\bar{X}_s-\rho\ddot{\bar{u}}_s = 0 \qquad (13)$$

$$\bar{p}_s+\Delta p_s = [\sigma_{ij}\bar{u}_{s,j}+\bar{\sigma}_{ij}(\delta_{js}+u_{s,j}+\bar{u}_{s,j})]n_i \qquad (14)$$

A theory akin to classical conservative and nonconservative buckling is of interest here. It is assumed that the perturbation is so small that the product $\bar{\sigma}_{ij}\bar{u}_{s,j}$ may be neglected, thus linearizing the equations, and the initial displacement gradients are also so small that the terms $\bar{\sigma}_{ij}u_{s,j}$ may be dropped which further simplifies the governing equations. This results in the following set of simplified linear equations:

$$(\sigma_{ij}\bar{u}_{s,j})_{,i}+\bar{\sigma}_{is,i}+\bar{X}_s+\Delta X_s = \rho\ddot{\bar{u}}_s \qquad (15)$$

$$\bar{p}_s+\Delta p_s = (\sigma_{ij}\bar{u}_{s,j}+\bar{\sigma}_{is})n_i \qquad (16)$$

IV. Initially Stressed Plate

The incremental displacements are assumed to be of the following form:

$$\bar{u}_1(x_1, x_2, x_3, t) = u_1(x_1, x_2, t)+x_3\psi_1(x_1, x_2, t) \qquad (17)$$

$$\bar{u}_2(x_1, x_2, x_3, t) = u_2(x_1, x_2, t)+x_3\psi_2(x_1, x_2, t) \qquad (18)$$

$$\bar{u}_3(x_1, x_2, t) = w(x_1, x_2, t) \qquad (19)$$

Such a displacement field is like the one Mindlin[4] used in his derivation of the equations for a noninitially stressed thick plate. This field includes the effect of transverse shear. $\bar{u}_1$ and $\bar{u}_2$ are the in-plane displacements and $\bar{u}_3$ is the lateral deflection of the neutral surface. u_1, u_2, and w are displacements of the neutral surface. ψ_1 and ψ_2 account for the effect of transverse shear.

The incremental stress-displacement relations are taken to be those of linear elasticity

$$\bar{\sigma}_{11} = (E/1-\nu^2)(\bar{u}_{1,1}+\nu\bar{u}_{2,2}) \qquad (20)$$

$$\bar{\sigma}_{22} = (E/1-\nu^2)(\bar{u}_{2,2}+\nu\bar{u}_{1,1}) \qquad (21)$$

$$\bar{\sigma}_{12} = G(\bar{u}_{1,2}+\bar{u}_{2,1}) \qquad (22)$$

$$\bar{\sigma}_{13} = \kappa^2 G^*(\bar{u}_{1,3}+\bar{u}_{3,1}) \qquad (23)$$

$$\bar{\sigma}_{23} = \kappa^2 G^*(\bar{u}_{2,3}+\bar{u}_{3,2}) \qquad (24)$$

where G^* takes into account the effects of transverse isotropy. If $G^* = G$ the material is isotropic. Substituting Eqs. (17–19) into Eqs. (20–24) gives

$$\bar{\sigma}_{11} = (E/1-\nu^2)(u_{1,1}+x_3\psi_{1,1}+\nu u_{2,2}+\nu x_3\psi_{2,2}) \qquad (25)$$

$$\bar{\sigma}_{22} = (E/1-\nu^2)(u_{2,2}+x_3\psi_{2,2}+\nu u_{1,1}+\nu x_3\psi_{1,1}) \qquad (26)$$

$$\bar{\sigma}_{12} = G(u_{1,2}+x_3\psi_{1,2}+u_{2,1}+x_3\psi_{2,1}) \qquad (27)$$

$$\bar{\sigma}_{13} = \kappa^2 G^*(\psi_1+w_{,1}) \qquad (28)$$

$$\bar{\sigma}_{23} = \kappa^2 G^*(\psi_2+w_{,2}) \qquad (29)$$

In order to give clarity to the integration procedure it is useful to partially write out Eq. (15)

$$(\sigma_{ij}\bar{u}_{1,j})_{,i}+\bar{\sigma}_{i1,i}+\bar{X}_1+\Delta X_1 = \rho\ddot{\bar{u}}_1 \qquad (30)$$

$$(\sigma_{ij}\bar{u}_{2,j})_{,i}+\bar{\sigma}_{i2,i}+\bar{X}_2+\Delta X_2 = \rho\ddot{\bar{u}}_2 \qquad (31)$$

$$(\sigma_{ij}\bar{u}_{3,j})_{,i}+\bar{\sigma}_{i3,i}+\bar{X}_3+\Delta X_3 = \rho\ddot{\bar{u}}_3 \qquad (32)$$

Equations (30–32) are due, respectively, to the x_1, x_2, and x_3 components of the stresses and inertia terms. Thus, integrating Eq. (30) through the thickness of the plate will result in the force equation for extension in the x_1-direction. Similarly, integrating Eq. (31) through the thickness will result in the force equation for extension in the x_2-direction. Integrating Eq. (32) through the thickness results in the shear force equation in the x_3-direction. Multiplying Eq. (30) by x_3 and integrating through the thickness results in the x_1-moment equation. Finally, multiplying Eq. (31) by x_3 and integrating through the thickness results in the x_2-moment equation. Since there are five unknowns this procedure gives the five equations necessary for a well posed problem. Keeping the inertia terms in Eqs. (30) and Eq. (31) accounts for rotary and extensional inertia effects. This discussion is summarized below.

$$x_1\text{-extensional equation} \quad \int_{-h/2}^{h/2} \text{Eq. (30) } dx_3 \qquad (33)$$

$$x_2\text{-extensional equation} \quad \int_{-h/2}^{h/2} \text{Eq. (31) } dx_3 \qquad (34)$$

$$x_3\text{-shear equation} \quad \int_{-h/2}^{h/2} \text{Eq. (32) } dx_3 \qquad (35)$$

$$x_1\text{-moment equation} \quad \int_{-h/2}^{h/2} x_3 \text{ Eq. (30) } dx_3 \qquad (36)$$

$$x_2\text{-moment equation} \quad \int_{-h/2}^{h/2} x_3 \text{ Eq. (31) } dx_3 \qquad (37)$$

Before performing these integrations it will be helpful to define the following initial stress resultants and material parameters. All integrals are from $-h/2$ to $h/2$.

$$N_x = \int\sigma_{11}\,dx_3, \quad N_y = \int\sigma_{22}\,dx_3, \quad N_z = \int\sigma_{33}\,dx_3$$
$$N_{xy} = \int\sigma_{12}\,dx_3$$
$$Q_x = \int\sigma_{13}\,dx_3, \quad Q_y = \int\sigma_{23}\,dx_3, \quad Q_x^* = \int x_3\sigma_{13}\,dx_3$$
$$Q_y^* = \int x_3\sigma_{23}\,dx_3 \qquad (38)$$
$$M_x = \int\sigma_{11}x_3\,dx_3, \quad M_y = \int\sigma_{22}x_3\,dx_3, \quad M_{xy} = \int\sigma_{12}x_3\,dx_3$$
$$M_x^* = \int\sigma_{11}x_3^2\,dx_3, \quad M_y^* = \int\sigma_{22}x_3^2\,dx_3,$$
$$M_{xy}^* = \int\sigma_{12}x_3^2\,dx_3$$

$$D = Eh/(1-\nu^2), \quad \mathscr{D} = Eh^3/12(1-\nu^2), \quad Gh^3/12 = \mathscr{D}(1-\nu)/2 \qquad (39)$$

Perform the integrations indicated in Eqs. (33–37). Then use the stress-displacement relations [Eqs. (25–29)] and definitions [Eqs. (38–39)]. It is necessary to perform the integrations before using the stress-displacement relations in order to attach proper meaning to body force and lateral loading terms. This procedure gives the following governing equations for the incremental displacements.‡

The x_1-extension equation is

$$D(u_{1,1}+\nu u_{2,2})_{,1}+Gh(u_{1,2}+u_{2,1})_{,2}+ (N_x u_{1,1}+M_x\psi_{1,1}+N_{xy}u_{1,2}+M_{xy}\psi_{1,2}+\psi_1 Q_x)_{,1}+ (N_y u_{1,2}+M_y\psi_{1,2}+N_{xy}u_{1,1}+M_{xy}\psi_{1,1}+\psi_1 Q_y)_{,2}+f_x = \rho h\ddot{u}_1 \qquad (40a)$$

where

$$f_x=\int_{-h/2}^{h/2}(\bar{X}_1+\Delta X_1)\,dx_3+u_{1,1}(\sigma_{31}{}^+-\sigma_{31}{}^-)+ \frac{h}{2}\psi_{1,1}(\sigma_{31}{}^+ +\sigma_{31}{}^-)+u_{1,2}(\sigma_{32}{}^+-\sigma_{32}{}^-)+ \frac{h}{2}\psi_{1,2}(\sigma_{32}{}^+ +\sigma_{32}{}^-)+\psi_1(\sigma_{33}{}^+-\sigma_{33}{}^-)+\bar{\sigma}_{31}{}^+-\bar{\sigma}_{31}{}^- \qquad (40b)$$

The (+) and (−) imply that the stresses are evaluated at the top and bottom faces of the plate, respectively. The terms containing the initial stresses arise from the change in the initial load brought about by the incremental deformation. The barred stresses are due to the load causing the incremental deformation. Similar discussions hold for the remaining equations.

The x_2-extension equation is

$$Gh(u_{1,2}+u_{2,1})_{,1}+D(u_{2,2}+\nu u_{1,1})_{,2}+ (N_x u_{2,1}+M_x\psi_{2,1}+N_{xy}u_{2,2}+M_{xy}\psi_{2,2}+Q_x\psi_2)_{,1}+ (N_y u_{2,2}+M_y\psi_{2,2}+N_{xy}u_{2,1}+M_{xy}\psi_{2,1}+Q_y\psi_2)_{,2}+f_y = \rho h\ddot{u}_2 \qquad (41a)$$

where

$$f_y=\int_{-h/2}^{h/2}(\bar{X}_2+\Delta X_2)\,dx_3+u_{2,1}(\sigma_{31}{}^+-\sigma_{31}{}^-)+ \frac{h}{2}\psi_{2,1}(\sigma_{31}{}^+ +\sigma_{31}{}^-)+u_{2,2}(\sigma_{32}{}^+-\sigma_{32}{}^-)+ \frac{h}{2}\psi_{2,2}(\sigma_{32}{}^+ +\sigma_{32}{}^-)+\psi_2(\sigma_{33}{}^+-\sigma_{33}{}^-)+\bar{\sigma}_{32}{}^+-\bar{\sigma}_{32}{}^- \qquad (41b)$$

The x_3-shear force equation is

$$\kappa^2G^*h[(\psi_1+w_{,1})_{,1}+(\psi_2+w_{,2})_{,2}]+(N_x w_{,1}+N_{xy}w_{,2})_{,1}+ (N_{xy}w_{,1}+N_y w_{,2})_{,2}+q=\rho h\ddot{w} \qquad (42a)$$

where

$$q=\int_{-h/2}^{h/2}(\bar{X}_3+\Delta X_3)\,dx_3+w_{,1}(\sigma_{31}{}^+-\sigma_{31}{}^-)+ w_{,2}(\sigma_{32}{}^+-\sigma_{32}{}^-)+\bar{\sigma}_{33}{}^+-\bar{\sigma}_{33}{}^- \qquad (42b)$$

The x_1-moment equation is

$$\mathscr{D}(\psi_{1,1}+\nu\psi_{2,2})_{,1}+\frac{\mathscr{D}}{2}(1-\nu)(\psi_{1,2}+\psi_{2,1})_{,2}+ (M_x u_{1,1}+M_x{}^*\psi_{1,1}+M_{xy}u_{1,2}+M_{xy}{}^*\psi_{1,2}+Q_x{}^*\psi_1)_{,1}+ (M_{xy}u_{1,1}+M_{xy}{}^*\psi_{1,1}+M_y u_{1,2}+M_y{}^*\psi_{1,2}+Q_y{}^*\psi_1)_{,2}- Q_x u_{1,1}-Q_x{}^*\psi_{1,1}-Q_y u_{1,2}-Q_y{}^*\psi_{1,2}-N_z\psi_1- \kappa^2G^*h(\psi_1+w_{,1})+m_x=\frac{\rho h^3}{12}\ddot{\psi}_1 \qquad (43a)$$

where

$$m_x=\int_{-h/2}^{h/2}(\bar{X}_1+\Delta X_1)x_3\,dx_3+\frac{h}{2}\Big[u_{1,1}(\sigma_{31}{}^+ +\sigma_{31}{}^-)+ \frac{h}{2}\psi_{1,1}(\sigma_{31}{}^+-\sigma_{31}{}^-)+u_{1,2}(\sigma_{32}{}^+ +\sigma_{32}{}^-)+ \frac{h}{2}\psi_{1,2}(\sigma_{32}{}^+-\sigma_{32}{}^-)+\psi_1(\sigma_{33}{}^+ +\sigma_{33}{}^-)+\bar{\sigma}_{13}{}^+ +\bar{\sigma}_{13}{}^-\Big] \qquad (43b)$$

The x_2-moment equation is

$$\frac{\mathscr{D}}{2}(1-\nu)(\psi_{1,2}+\psi_{2,1})_{,1}+\mathscr{D}(\psi_{2,2}+\nu\psi_{1,1})_{,2}+ (M_x u_{2,1}+M_x{}^*\psi_{2,1}+M_{xy}u_{2,2}+M_{xy}{}^*\psi_{2,2}+Q_x{}^*\psi_2)_{,1}+ (M_{xy}u_{2,1}+M_{xy}{}^*\psi_{2,1}+M_y u_{2,2}+M_y{}^*\psi_{2,2}+Q_y{}^*\psi_2)_{,2}- Q_x u_{2,1}-Q_x{}^*\psi_{2,1}-Q_y u_{2,2}-Q_y{}^*\psi_{2,2}-N_z\psi_2- \kappa^2G^*h(\psi_2+w_{,2})+m_y=\rho(h^3/12)\ddot{\psi}_2 \qquad (44a)$$

where

$$m_y=\int_{-h/2}^{h/2}(\bar{X}_2+\Delta X_2)x_3\,dx_3+\frac{h}{2}\Big[(\sigma_{31}{}^+ +\sigma_{31}{}^-)u_{2,1}+ \frac{h}{2}(\sigma_{31}{}^+-\sigma_{31}{}^-)\psi_{2,1}+(\sigma_{32}{}^+ +\sigma_{32}{}^-)u_{2,2}+ \frac{h}{2}(\sigma_{32}{}^+-\sigma_{32}{}^-)\psi_{2,2}+(\sigma_{33}{}^+ +\sigma_{33}{}^-)\psi_2+\bar{\sigma}_{23}{}^+ +\bar{\sigma}_{23}{}^-\Big] \qquad (44b)$$

The boundary conditions on the edge of the plate are considered now. On the edge of a plate the n_3 component of the unit normal is zero. If Eq. (16) is phrased in terms of stresses normal and tangent to the edge then $n_n=1$, $n_t=0$ and $n_3=0$ on the edge. Thus

$$\bar{p}_n+\Delta p_n=\sigma_{nn}\bar{u}_{n,n}+\sigma_{nt}\bar{u}_{n,t}+\sigma_{n3}\bar{u}_{n,3}+\bar{\sigma}_{nn} \qquad (45)$$

$$\bar{p}_t+\Delta p_t=\sigma_{nn}\bar{u}_{t,n}+\sigma_{nt}\bar{u}_{t,t}+\sigma_{n3}\bar{u}_{t,3}+\bar{\sigma}_{nt} \qquad (46)$$

$$p_3+\Delta p_3=\sigma_{nn}\bar{u}_{3,n}+\sigma_{nt}\bar{u}_{3,t}+\sigma_{n3}\bar{u}_{3,3}+\bar{\sigma}_{n3} \qquad (47)$$

where the subscripts n and t denote the normal and tangential directions of the plate's edge, respectively. The terms containing the initial stresses account for the change in the initial boundary conditions due to the incremental deformation.

It is convenient to define the following resultants where all integrals are from $-h/2$ to $h/2$.

$$\begin{aligned} \Delta F_{nn}&=\int\Delta p_n\,dx_3 & \bar{F}_{nn}&=\int\bar{p}_n\,dx_3\\ \Delta F_{nt}&=\int\Delta p_t\,dx_3 & \bar{F}_{nt}&=\int\bar{p}_t\,dx_3\\ \Delta F_{n3}&=\int\Delta p_3\,dx_3 & \bar{F}_{n3}&=\int\bar{p}_3\,dx_3\\ \Delta M_{nn}&=\int\Delta p_n x_3\,dx_3 & \bar{M}_{nn}&=\int\bar{p}_n x_3\,dx_3\\ \Delta M_{nt}&=\int\Delta p_t x_3\,dx_3 & \bar{M}_{nt}&=\int\bar{p}_t x_3\,dx_3 \end{aligned} \qquad (48)$$

Integrating Eqs. (45–47) through the thickness yields the boundary force conditions and integrating Eqs. (45) and (46) multiplied by x_3 yields the boundary moment conditions. Then, using the definitions (38) put in terms of normal and tangential coordinates and applying the stress-displacement relations (25–29) yields the following five boundary conditions:

$$\bar{F}_{nn}+\Delta F_{nn}=N_n u_{n,n}+M_n\psi_{n,n}+N_{nt}u_{n,t}+M_{nt}\psi_{n,t}+Q_n\psi_n+ D(u_{n,n}+\nu u_{t,t}) \qquad (49)$$

$$\bar{F}_{nt}+\Delta F_{nt}=N_n u_{t,n}+M_n\psi_{t,n}+N_{nt}u_{t,t}+M_{nt}\psi_{t,t}+Q_n\psi_t+ Gh(u_{n,t}+u_{t,n}) \qquad (50)$$

$$\bar{F}_{n3}+\Delta F_{n3}=N_n w_{,n}+N_{nt}w_{,t}+\kappa^2G^*h(\psi_n+w_{,n}) \qquad (51)$$

$$\bar{M}_{nn}+\Delta M_{nn}=M_n u_{n,n}+M_n{}^*\psi_{n,n}+M_{nt}u_{n,t}+M_{nt}{}^*\psi_{n,t}+ Q_n{}^*\psi_n+\mathscr{D}(\psi_{n,n}+\nu\psi_{t,t}) \qquad (52)$$

$$\bar{M}_{nt}+\Delta M_{nt}=M_n u_{t,n}+M_n{}^*\psi_{t,n}+M_{nt}u_{t,t}+M_{nt}{}^*\psi_{t,t}+ Q_n{}^*\psi_t+\frac{\mathscr{D}}{2}(1-\nu)(\psi_{n,t}+\psi_{t,n}) \qquad (53)$$

Alternative displacement boundary conditions are

$$\left.\begin{aligned} u_n&=u_{nn}\\ u_t&=u_{nt}\\ w&=w_{n3}\\ \psi_n&=\psi_{nn}\\ \psi_t&=\psi_{nt} \end{aligned}\right\} \qquad (54)$$

where the quantities on the right-hand side are prescribed.

If a rectangular plate were being considered the boundary conditions would be rephrased in x_1, x_2 coordinates. For

‡ Note that the incremental displacements as defined by Eqs. (17–19) do not have a superior bar. These displacements should not be confused with the initial displacements of Eq. (7).

example, on the edge x_2 is constant the unit normal has components (0, 1, 0) and the subscript n corresponds to the x_2-direction while the subscript t corresponds to the x_1-direction. Thus, for instance, Eq. (53) would become

$$\bar{M}_{21}+\Delta M_{21} = M_y u_{1,2}+M^*\psi_{1,2}+M_{yx}u_{1,1}+M_{yx}{}^*\psi_{1,1}+ Q_y{}^*\psi_1+\frac{\mathscr{D}}{2}(1-\nu)(\psi_{2,1}+\psi_{1,2})$$

The plate equations derived here differ from those derived in Ref. 5 where a variational procedure was used. In order to have more confidence in the present equations they will be rederived in the next section using the variational method.

V. Variational Derivation of the Plate Equations

Hamilton's principle is used to derive the nonlinear plate equations which will include the effects of rotary inertia and transverse shear. The resulting equations are then perturbed by an incremental deformation. The resulting equations are then linearized to obtain the desired governing equations. The procedure is the same as that used in Ref. 5 except that no terms are dropped, explicit expressions are given for lateral loading terms and Trefftz stresses are used throughout rather than a mixture of stresses.

In deriving nonlinear equations using a variational procedure care must be taken to use definitions of stress and strain that are compatible. This is because stress and strain can be defined in several ways and only certain combinations are physically meaningful for a variational procedure. This problem does not arise in the linear theory because there is no difference between material (Lagrangian) and spatial (Eulerian) descriptions of strains and stresses taken with respect to initial areas and stresses taken with respect to final areas are identical. Further, in the nonlinear theory a stress vector may be decomposed along coordinate curves using either unit vectors or lattice vectors, a choice that normally does not occur in the linear theory.

Hamilton's principle may be expressed as

$$\delta\int_{t_o}^{t_1}(U-K-W_e-W_i)\,dt = 0 \tag{55}$$

where U is the strain energy, K the kinetic energy, W_e the work of external forces, and W_i the work of internal forces. Taking Φ as the strain energy per unit initial volume it can be shown[1,2] that a compatible combination of stress and strain is the Trefftz stresses and material (Lagrangian) strains. Thus

$$\begin{aligned} U &= \textstyle\int_{V_o}\Phi\,dV \\ K &= \tfrac{1}{2}\textstyle\int_{V_o}\rho\dot{v}_i\dot{v}_i\,dV \\ W_e &= \textstyle\int_{S_o}p_i{}^*v_i\,dS \\ W_i &= \textstyle\int_{V_o}X_i{}^*v_i\,dV \end{aligned} \tag{56}$$

also

$$\delta U = \textstyle\int_{V_o}\delta\Phi\,dV = \int_{V_o}\sigma_{ij}{}^*\,\delta e_{ij}\,dV \tag{57}$$

where the $\sigma_{ij}{}^*$ are Trefftz stress components referred to the material coordinates, the e_{ij} are material strains referred to material coordinates, the v_i are Cartesian displacement components referred to the spatial frame, the $p_i{}^*$ are Cartesian components of the surface force per unit initial area, the $X_i{}^*$ are Cartesian body force components per unit initial volume, ρ is the initial mass density, V_o the initial volume, and S_o the initial boundary surface. Substituting Eqs. (56) and (57) into Hamilton's principle, Eqs. (55), yields

$$\delta\int_{t_o}^{t_1}\left[\int_{V_o}(\Phi-X_i{}^*v_i-\tfrac{1}{2}\rho\dot{v}_i\dot{v}_i)\,dV-\int_{S_o}p_i{}^*v_i\,dS\right]dt \tag{58}$$

Assume that the stresses and applied forces do not vary. Then taking the variation, integrating the kinetic energy term by parts with respect to time and using the assumption that δv_i vanishes at t_o and t_1 Eq. (58) becomes

$$\int_{t_o}^{t_1}\left[\int_{V_o}(\sigma_{ij}{}^*\delta e_{ij}-X_i{}^*\delta v_i+\rho\ddot{v}_i\,\delta v_i)\,dV-\int_{S_o}p_i{}^*\delta v_i\,dS\right]dt = 0 \tag{59}$$

The Lagrangian strain components are related to the displacements by

$$e_{ij} = \tfrac{1}{2}(v_{i,j}+v_{j,i}+v_{k,i}\,v_{k,j}) \tag{60}$$

Assume the same form for the displacements as was used in Eqs. (17–19).

$$v_1(x_1,x_2,x_3,t) = u_1(x_1,x_2,t)+x_3\psi_1(x_1,x_2,t) \tag{61}$$

$$v_2(x_1,x_2,x_3,t) = u_2(x_1,x_2,t)+x_3\psi_2(x_1,x_2,t) \tag{62}$$

$$v_3(x_1,x_2,x_3,t) = w(x_1,x_2,t) \tag{63}$$

Since the time integral in Eq. (59) is over an arbitrary time interval the term in brackets must vanish. Substitute Eqs. (60–63) into Eq. (59). In the volume integral perform the integration with respect to x_3 through the thickness of the plate from $-h/2$ to $h/2$. Then define stress resultants in the same way as in Eqs. (38). Finally, perform all necessary partial integrations needed to remove derivatives from the variations of the displacements and collect terms that contain variations of the same displacements to find

$$\begin{aligned}
&-\textstyle\int\int\{[(N_x+N_xu_{1,1}+M_x\psi_{1,1})_{,1}+(N_yu_{1,2}+M_y\psi_{1,2})_{,2}+ \\
&(N_{xy}+N_{xy}u_{1,1}+M_{xy}\psi_{1,1})_{,2}+(N_{xy}u_{1,2}+M_{xy}\psi_{1,2})_{,1}+ \\
&(Q_x\psi_1)_{,1}+(Q_y\psi_1)_{,2}]\,\delta u_1+[(M_x+M_xu_{1,1}+M_x{}^*\psi_{1,1}+ \\
&M_{xy}u_{1,2}+M_{xy}{}^*\psi_{1,2}+Q_x{}^*\psi_1)_{,1}+(M_yu_{1,2}+M_y{}^*\psi_{1,2}+ \\
&M_{xy}+M_{xy}u_{1,1}+M_{xy}{}^*\psi_{1,1}+Q_y{}^*\psi_1)_{,2}-Q_x-Q_xu_{1,1}- \\
&Q_x{}^*\psi_{1,1}-Q_yu_{1,2}-Q_y{}^*\psi_{1,2}-N_z\psi_1]\,\delta\psi_1+[(N_xu_{2,1}+ \\
&M_x\psi_{2,1}+N_{xy}+N_{xy}u_{2,2}+M_{xy}\psi_{2,2}+Q_x\psi_2)_{,1}+ \\
&(N_y+N_yu_{2,2}+M_y\psi_{2,2}+N_{xy}u_{2,1}+M_{xy}\psi_{2,1}+ \\
&Q_y\psi_2)_{,2}]\,\delta u_2+[(M_xu_{2,1}+M_x{}^*\psi_{2,1}+M_{xy}+M_{xy}u_{2,2}+ \\
&M_{xy}{}^*\psi_{2,2}+Q_x{}^*\psi_2)_{,1}+(M_y+M_yu_{2,2}+M_y{}^*\psi_{2,2}+ \\
&M_{xy}u_{2,1}+M_{xy}{}^*\psi_{2,1}+Q_y{}^*\psi_2)_{,2}-Q_xu_{2,1}-Q_x{}^*\psi_{2,1}- \\
&Q_y-Q_yu_{2,2}-Q_y{}^*\psi_{2,2}-N_z\psi_2]\,\delta\psi_2+[(N_xw_{,1}+ \\
&N_{xy}w_{,2}+Q_x)_{,1}+(N_yw_{,2}+N_{xy}w_{,1}+Q_y)_{,2}]\,\delta w- \\
&\int_{-h/2}^{h/2}(X_1{}^*\delta u_1+x_3X_1{}^*\delta\psi_1+X_2{}^*\delta u_2+x_3X_2{}^*\delta\psi_2+ \\
&X_3{}^*\delta w)\,dx_3+\rho h\ddot{u}_1\delta u_1+\frac{\rho h^3}{12}\ddot{\psi}_1\delta\psi_1+\rho h\ddot{u}_2\delta u_2+ \\
&\left.\frac{\rho h^3}{12}\ddot{\psi}_2\delta\psi_2+\rho h\ddot{w}\,\delta w\right\}dx_2\,dx_3+\textstyle\int[(N_{xy}+N_{xy}u_{1,1}+ \\
&M_{xy}\psi_{1,1}+N_yu_{1,2}+M_y\psi_{1,2}+Q_y\psi_1)\,\delta u_1+(M_{xy}+ \\
&M_{xy}u_{1,1}+M_{xy}{}^*\psi_{1,1}+M_yu_{1,2}+M_y{}^*\psi_{1,2}+Q_y{}^*\psi_1)\,\delta\psi_1+ \\
&(N_y+N_yu_{2,2}+M_y\psi_{2,2}+N_{xy}u_{2,1}+M_{xy}\psi_{2,1}+ \\
&Q_y\psi_2)\,\delta u_2+(M_y+M_yu_{2,2}+M_y{}^*\psi_{2,2}+M_{xy}u_{2,1}+ \\
&M_{xy}{}^*\psi_{2,1}+Q_y{}^*\psi_2)\,\delta\psi_2+(N_yw_{,2}+N_{xy}w_{,1}+Q_y)\,\delta w]\,dx_1+ \\
&\textstyle\int[(N_x+N_xu_{1,1}+M_x\psi_{1,1}+N_{xy}u_{1,2}+M_{xy}\psi_{1,2}+ \\
&Q_x\psi_1)\,\delta u_1+(M_x+M_xu_{1,1}+M_x{}^*\psi_{1,1}+M_{xy}u_{1,2}+ \\
&M_{xy}{}^*\psi_{1,2}+Q_x{}^*\psi_1)\,\delta\psi_1+(N_{xy}+N_{xy}u_{2,2}+M_{xy}\psi_{2,2}+ \\
&N_xu_{2,1}+M_x\psi_{2,1}+Q_x\psi_2)\,\delta u_2+(M_{xy}+M_{xy}u_{2,2}+ \\
&M_{xy}{}^*\psi_{2,2}+M_xu_{2,1}+M_x{}^*\psi_{2,1}+Q_x{}^*\psi_2)\,\delta\psi_2+ \\
&(N_{xy}w_{,2}+N_xw_{,1}+Q_x)\,\delta w]\,dx_2-\textstyle\int_{S_o}p_i{}^*\,\delta u_i\,dS = 0
\end{aligned} \tag{64}$$

In the last integral over the bounding surface no substitutions have been made as yet. This integral may be written as a sum of integrals over the bounding edge and on the faces of the plate.

$$\int_{S_o}p_i{}^*\delta v_i\,dS = \oint_{C_o}\int_{-h/2}^{h/2}p_i{}^*\delta v_i\,dx\,dC+\iint_{A_o{}^+}p_i{}^*\delta v_i\,dx_2\,dx_3-\iint_{A_o{}^-}p_i{}^*\delta v_i\,dx_2\,dx_3 \tag{65}$$

where the first integral is over the edge and the last two are over the top and bottom faces, respectively. Consider the integral around the boundary edge first and use Eqs. (61–63) to find

$$\oint_{C_o}\int_{-h/2}^{h/2} p_i{}^*\delta v_i\, dx_3\, dC = \oint_{C_o}\int_{-h/2}^{h/2} [p_1{}^*(\delta u_1 + x_3\,\delta\psi_1) + p_2{}^*(\delta u_2 + x_3\,\delta\psi_2) + p_3{}^*\delta w]\, dx_3\, dC \qquad (66)$$

In the last two integrals in Eq. (65), Eq. (4) may be used in order to find explicit relations for the lateral loads. On these faces the components of the unit normal are (0, 0, 1). Thus

$$\iint_{A_o^+} p_i{}^*\delta v_i\, dx_2\, dx_3 - \iint_{A_o^-} p_i{}^*\delta v_i\, dx_3\, dx_3 = \iint p_i{}^*\delta v_i\, dx_2\, dx_3\Big|_{z=-h/2}^{z=h/2} = \iint \{[\sigma_{31}{}^*(1+u_{1,1}+x_3\psi_{1,1}) + \sigma_{32}{}^*(u_{1,2}+x_3\psi_{1,2}) + \sigma_{33}{}^*\psi_1](\delta u_1 + x_3\delta\psi_1) + [\sigma_{31}{}^*(u_{2,1}+x_3\psi_{2,1}) + \sigma_{32}{}^*(1+u_{2,2}+x_3\psi_{2,2}) + \sigma_{33}{}^*\psi_2](\delta u_2 + x_3\delta\psi_2) + [\sigma_{31}{}^*w_{,1} + \sigma_{32}{}^*w_{,2} + \sigma_{33}{}^*]\,\delta w\} \times dx_2\, dx_3\Big|_{-h/2}^{h/2} \qquad (67)$$

Equations (66) and (67) are substituted back into Eq. (65) which is subsequently substituted back into Eq. (64). The integrals around the edge can be phrased in terms of normal and tangential coordinates. Use is then made of the fact that integrals on the edge vanish separately from the area integrals. Finally, because the displacements are independent and their variations are arbitrary the following nonlinear equations for a plate including the transverse shear and rotary inertia effects are found together with the appropriate boundary conditions.

The x_1-extension equation is

$$(N_x + N_x u_{1,1} + M_x\psi_{1,1} + N_{xy}u_{1,2} + M_{xy}\psi_{1,2} + Q_x\psi_1)_{,1} + (N_{xy} + N_{xy}u_{1,1} + M_{xy}\psi_{1,1} + N_y u_{1,2} + M_y\psi_{1,2} + Q_y\psi_1)_{,2} + f_x = \rho h\ddot{u}_1 \qquad (68a)$$

where

$$f_x = \int_{-h/2}^{h/2} X_1{}^*\, dx_3 + (\sigma_{31}{}^{*+} - \sigma_{31}{}^{*-})(1+u_{1,1}) + \frac{h}{2}(\sigma_{31}{}^{*+} + \sigma_{31}{}^{*-})\psi_{1,1} + (\sigma_{32}{}^{*+} - \sigma_{32}{}^{*-})u_{1,2} + \frac{h}{2}(\sigma_{32}{}^{*+} + \sigma_{32}{}^{*-})\psi_{1,2} + (\sigma_{33}{}^{*+} - \sigma_{33}{}^{*-})\psi_1 \qquad (68b)$$

where the terms containing the stresses have come from evaluating the appropriate terms in Eq. (67).

The x_2-extension equation is

$$(N_{xy} + N_{xy}u_{2,2} + M_{xy}\psi_{2,2} + N_x u_{2,1} + M_x\psi_{2,1} + Q_x\psi_2)_{,1} + (N_y + N_y u_{2,2} + M_y\psi_{2,2} + N_{xy}u_{2,1} + M_{xy}\psi_{2,1} + Q_y\psi_2)_{,2} + f_y = \rho h\ddot{u}_2 \qquad (69a)$$

where

$$f_y = \int_{-h/2}^{h/2} X_2{}^*\, dx_2 + (\sigma_{32}{}^{*+} - \sigma_{32}{}^{*-})(1+u_{2,2}) + \frac{h}{2}(\sigma_{32}{}^{*+} + \sigma_{32}{}^{*-})\psi_{2,2} + (\sigma_{31}{}^{*+} - \sigma_{31}{}^{*-})u_{2,1} + \frac{h}{2}(\sigma_{31}{}^{*+} + \sigma_{31}{}^{*-})\psi_{2,1} + (\sigma_{33}{}^{*+} - \sigma_{33}{}^{*-})\psi_2 \qquad (69b)$$

The shear force equation is

$$(N_x w_{,1} + N_{xy}w_{,2} + Q_x)_{,1} + (N_y w_{,2} + N_{xy}w_{,1} + Q_y)_{,2} + f_z = \rho h\ddot{w} \qquad (70a)$$

where

$$f_z = \int_{-h/2}^{h/2} X_3{}^*\, dx_3 + (\sigma_{31}{}^{*+} - \sigma_{31}{}^{*-})w_{,1} + (\sigma_{32}{}^{*+} - \sigma_{32}{}^{*-})w_{,2} + (\sigma_{33}{}^{*+} - \sigma_{33}{}^{*-}) \qquad (70b)$$

The x_1-moment equation is

$$(M_x + M_x u_{1,1} + M_x{}^*\psi_{1,1} + M_{xy}u_{1,2} + M_{xy}{}^*\psi_{1,2} + Q_x{}^*\psi_1)_{,1} + (M_{xy} + M_{xy}u_{1,1} + M_{xy}{}^*\psi_{1,1} + M_y u_{1,2} + M_y{}^*\psi_{1,2} + Q_y{}^*\psi_1)_{,2} - Q_x - Q_x u_{1,1} - Q_x{}^*\psi_{1,1} - Q_y u_{1,2} - Q_y{}^*\psi_{1,2} - N_z\psi_1 + m_x = \frac{\rho h^3}{12}\ddot{\psi}_1 \qquad (71a)$$

where

$$m_x = \int_{-h/2}^{h/2} x_3 X_1{}^*\, dx + \frac{h}{2}(\sigma_{31}{}^{*+} + \sigma_{31}{}^{*-})(1+u_{1,1}) + \frac{h^2}{4}(\sigma_{31}{}^{*+} - \sigma_{31}{}^{*-})\psi_{1,1} + \frac{h}{2}(\sigma_{32}{}^{*+} + \sigma_{32}{}^{*-})u_{1,2} + \frac{h^2}{4}(\sigma_{32}{}^{*+} - \sigma_{32}{}^{*-})\psi_{1,2} + \frac{h}{2}(\sigma_{33}{}^{*+} + \sigma_{33}{}^{*-})\psi_1 \qquad (71b)$$

The x_2-moment equation is

$$(M_{xy} + M_{xy}u_{2,2} + M_{xy}{}^*\psi_{2,2} + M_x u_{2,1} + M_x{}^*\psi_{2,1} + Q_x{}^*\psi_2)_{,1} + (M_y + M_y u_{2,2} + M_y{}^*\psi_{2,2} + M_{xy}u_{2,1} + M_{xy}{}^*\psi_{2,1} + Q_y{}^*\psi_2)_{,2} - Q_y - Q_y u_{2,2} - Q_y{}^*\psi_{2,2} - Q_x u_{2,1} - Q_x{}^*\psi_{2,1} - N_z\psi_2 + m_y = \frac{\rho h^3}{12}\ddot{\psi}_2 \qquad (72a)$$

where

$$m_y = \int_{-h/2}^{h/2} x_3 X_2{}^*\, dx_3 + \frac{h}{2}(\sigma_{32}{}^{*+} + \sigma_{32}{}^{*-})(1+u_{2,2}) + \frac{h^2}{4}(\sigma_{32}{}^{*+} - \sigma_{32}{}^{*-})\psi_{2,2} + \frac{h}{2}(\sigma_{31}{}^{*+} + \sigma_{31}{}^{*-})u_{2,1} + \frac{h^2}{4}(\sigma_{31}{}^{*+} - \sigma_{31}{}^{*-})\psi_{2,1} + \frac{h}{2}(\sigma_{33}{}^{*+} + \sigma_{33}{}^{*-})\psi_2 \qquad (72b)$$

The boundary conditions in terms of normal and tangential coordinates are

$$\left.\begin{aligned}
F_{nn} &= N_n + N_n u_{n,n} + M_n\psi_{n,n} + N_{nt}u_{n,t} + M_{nt}\psi_{n,t} + Q_n\psi_n \\
F_{nt} &= N_{nt} + N_{nt}u_{t,t} + M_{nt}\psi_{t,t} + N_n u_{t,n} + M_n\psi_{t,n} + Q_n\psi_t \\
F_{n3} &= N_n w_{,n} + N_{nt}w_{,t} + Q_n \\
L_{nn} &= M_n + M_n u_{n,n} + M_n{}^*\psi_{n,n} + M_{nt}u_{n,t} + M_{nt}{}^*\psi_{n,t} + Q_n{}^*\psi_n \\
L_{nt} &= M_{nt} + M_{nt}u_{t,t} + M_{nt}{}^*\psi_{t,t} + M_n u_{t,n} + M_n{}^*\psi_{t,n} + Q_n{}^*\psi_t
\end{aligned}\right\} \qquad (73a)$$

where

$$\left.\begin{aligned}
F_{nn} &= \int_{-h/2}^{h/2} p_n{}^*\, dx_3 \quad F_{nt} = \int_{-h/2}^{h/2} p_t{}^*\, dx_3 \quad F_{n3} = \int_{-/2}^{h2} p_3{}^*dx_3 \\
L_{nn} &= \int_{-h/2}^{h/2} x_3 p_n{}^*\, dx_3 \qquad L_{nt} = \int_{-h/2}^{h/2} x_3 p_t{}^*\, dx_3
\end{aligned}\right\} \qquad (73b)$$

Either the force conditions [Eq. (73)] or the respective displacement conditions

$$\left.\begin{aligned}
u_n &= u_{nn} \\
u_t &= u_{nt} \\
w &= w_{n3} \\
\psi_n &= \psi_{nn} \\
\psi_t &= \psi_{nt}
\end{aligned}\right\} \qquad (74)$$

are specified. The terms on the right-hand side of Eqs. (74) are given.

Equations (68–74) are the nonlinear governing equations for a plate including the effects of transverse shear and rotary inertia. The equations differ from those in Ref. 5 in that terms containing $M_x{}^*$, $M_y{}^*$, $M_{xy}{}^*$, $Q_x{}^*$, $Q_y{}^*$, and N_z are included here. Also, specific relationships are given for the lateral loading terms f_x, f_y, f_z, m_x, and m_y which are not given in Ref. 5.

In order to obtain the desired linear equations for a plate in a state of initial stress it is necessary to introduce the following perturbed quantities:

$$\hat{u}_1 = u_1 + \bar{u}_1, \quad \hat{u}_2 = u_2 + \bar{u}_2, \quad \hat{w} = w + \bar{w}$$
$$\hat{\sigma}_{ij}{}^* = \sigma_{ij}{}^* + \bar{\sigma}_{ij}{}^*, \quad p_i{}^* = p_i{}^* + \Delta p_i{}^* + \bar{p}_i{}^*, \qquad (75)$$
$$X_i{}^* = X_i{}^* + \Delta X_i{}^* + \bar{X}_i{}^*$$

Also

$$\ddot{u}_1 = \ddot{u}_2 = \ddot{w}_3 = \ddot{\psi}_1 = \ddot{\psi}_3 = 0$$

Introduce stress resultants of the form

$$\hat{N}_x = N_x + \bar{N}_x \qquad (76)$$

with similar expressions for the remaining N's, M's, and Q's. These are defined as in Eq. (38), e.g.,

$$\hat{N}_x = \int_{-h/2}^{h/2} (\sigma_{11}{}^* + \bar{\sigma}_{11}{}^*)\, dx_3$$

Further, the unbarred quantities are initial quantities which satisfy the equilibrium equations [i.e., Eqs. (68–72) without inertia terms]. The barred quantities are the incremental quantities and $\Delta X_i{}^*$ and $\Delta p_i{}^*$ are the changes in the initial applied forces due to the incremental deformation.

The perturbed quantities satisfy the equations of motion and boundary conditions (68–74). Thus, Eqs. (75) and (76) are substituted into these equations. Then the fact that the initial variables satisfy the equilibrium equations is used to finally yield nonlinear equations for the determination of the incremental variables. If extensions and shears are small the stresses, stress resultants, and applied loads take on the usual definitions in that initial and final areas and volumes are identical. The incremental stresses are assumed to satisfy the stress displacement relations [Eqs. (25–29)]. These are employed to replace the incremental stress resultants by displacements. Finally, the equations are linearized and simplified in the same way as Eqs. (13) and (14) were by dropping products of incremental stress resultants with initial and incremental displacement gradients.

There is no point in carrying this procedure through here as the resulting equations turn out to be the same as Eqs. (40–54).

VI. Initially Stressed Classical Plate

The equations for an initially stressed Mindlin plate may be reduced to those for an initially stressed thin plate with rotary inertia by making the appropriate assumptions. Thus, the following change to the displacement functions is made:

$$\psi_1 = -w_{,1} \qquad \psi_2 = -w_{,2} \tag{77}$$

The stress displacement relations for the incremental stresses now are

$$\begin{aligned}
\bar{\sigma}_{11} &= (E/1-\nu^2)(u_{1,1} - x_3 w_{,11} + \nu u_{2,2} - \nu x_3 w_{,22})\\
\bar{\sigma}_{12} &= G(u_{1,2} + u_{2,1} - 2x_3 w_{,12})\\
\bar{\sigma}_{22} &= (E/1-\nu^2)(u_{2,2} - x_3 w_{,22} + \nu u_{1,1} - \nu x_3 w_{,11})
\end{aligned} \tag{78}$$

with $\bar{\sigma}_{13}$, $\bar{\sigma}_{23}$, and $\bar{\sigma}_{33}$ unspecified. Since $\bar{\sigma}_{13}$ and $\bar{\sigma}_{23}$ were specified for the initially stressed Mindlin plate the following changes must be made in Eqs. (42–44) for use here

$$\begin{aligned}
\kappa^2 G^* h(\psi_1 + w_{,1}) &= \bar{Q}_x = \int_{-h/2}^{h/2} \bar{\sigma}_{13}\, dx_3\\
\kappa^2 G^* h(\psi_2 + w_{,2}) &= \bar{Q}_y = \int_{-h/2}^{h/2} \bar{\sigma}_{23}\, dx_3
\end{aligned} \tag{79}$$

The equations for the initially stressed classical plate are obtained by substituting Eqs. (77) and (79) into Eqs. (40–44). Then $\bar{Q}_x$ and $\bar{Q}_y$ (which are unspecified) are eliminated between Eqs. (42–44) to obtain the desired result.

$$\begin{aligned}
&D(u_{1,1} + \nu u_{2,2})_{,1} + \tfrac{1}{2}D(1-\nu)(u_{1,2} + u_{2,1})_{,2} +\\
&\quad (u_{1,1}N_x - w_{,11}M_x + u_{1,2}N_{xy} - w_{,12}M_{xy} - w_{,1}Q_x)_{,1} +\\
&\quad (u_{1,1}N_{xy} - w_{,11}M_{xy} + u_{1,2}N_y - w_{,12}M_y - w_{,1}Q_y)_{,2} +\\
&\qquad f_x = \rho h \ddot{u}_1
\end{aligned} \tag{80a}$$

where

$$\begin{aligned}
f_x = &\int_{-h/2}^{h/2} (\bar{X}_1 + \Delta X_1)\, dx_3 + u_{1,1}(\sigma_{31}{}^+ - \sigma_{31}{}^-) -\\
&\frac{h}{2} w_{,11}(\sigma_{31}{}^+ + \sigma_{31}{}^-) + u_{1,2}(\sigma_{32}{}^+ - \sigma_{32}{}^-) -\\
&\frac{h}{2} w_{,12}(\sigma_{32}{}^+ + \sigma_{32}{}^-) - w_{,1}(\sigma_{33}{}^+ - \sigma_{33}{}^-) +\\
&\bar{\sigma}_{31}{}^+ - \bar{\sigma}_{31}{}^-
\end{aligned} \tag{80b}$$

$$\begin{aligned}
&\tfrac{1}{2}D(1-\nu)(u_{1,2} + u_{2,1})_{,1} + D(u_{2,2} + \nu u_{1,1})_{,2} +\\
&\quad (u_{2,1}N_x - w_{,12}M_x + u_{2,2}N_{xy} - w_{,22}M_{xy} - w_{,2}Q_x)_{,1} +\\
&\quad (u_{2,1}N_{xy} - w_{,12}M_{xy} + u_{2,2}N_y - w_{,22}M_y - w_{,2}Q_y)_{,2} +\\
&\qquad f_y = \rho h \ddot{u}_2
\end{aligned} \tag{81a}$$

where

$$\begin{aligned}
f_y = &\int_{-h/2}^{h/2} (\bar{X}_2 + \Delta X_2)\, dx_3 + u_{2,1}(\sigma_{31}{}^+ - \sigma_{31}{}^-) -\\
&\frac{h}{2} w_{,21}(\sigma_{31}{}^+ + \sigma_{31}{}^-) + u_{2,2}(\sigma_{32}{}^+ - \sigma_{32}{}^-) -\\
&\frac{h}{2} w_{,22}(\sigma_{32}{}^+ + \sigma_{32}{}^-) - w_{,2}(\sigma_{33}{}^+ - \sigma_{33}{}^-) +\\
&\bar{\sigma}_{32}{}^+ - \bar{\sigma}_{32}{}^-
\end{aligned} \tag{81b}$$

$$\begin{aligned}
&\Big[w_{,1}N_x + w_{,2}N_{xy} + \frac{\rho h^3}{12}\ddot{w}_{,1} + m_x - \mathscr{D}(w_{,11} + w_{,22})_{,1} +\\
&\quad (u_{1,1}M_x - w_{,11}M_x{}^* + u_{1,2}M_{xy} - w_{,12}M_{xy}{}^* - w_{,1}Q_x{}^*)_{,1} +\\
&\quad (u_{1,1}M_{xy} - w_{,11}M_{xy}{}^* + u_{1,2}M_y - w_{,12}M_y{}^* - w_{,1}Q_y{}^*)_{,2} -\\
&\quad u_{1,1}Q_x + w_{,11}Q_x{}^* - u_{1,2}Q_y + w_{,12}Q_y{}^* + w_{,1}N_z\Big]_{,1} +\\
&\Big[w_{,1}N_{xy} + w_{,2}N_y + \frac{\rho h^3}{12}\ddot{w}_{,2} + m_y - \mathscr{D}(w_{,11} + w_{,22})_{,2} +\\
&\quad (u_{2,1}M_x - w_{,12}M_x{}^* + u_{2,2}M_{xy} - w_{,22}M_{xy}{}^* - w_{,2}Q_x{}^*)_{,1} +\\
&\quad (u_{2,1}M_{xy} - w_{,12}M_{xy}{}^* + u_{2,2}M_y - w_{,22}M_y{}^* - w_{,2}Q_y{}^*)_{,2} -\\
&\quad u_{2,1}Q_x + w_{,12}Q_x{}^* - u_{2,2}Q_y + w_{,22}Q_y{}^* + w_{,2}N_z\Big]_{,2} + q = \rho h \ddot{w}
\end{aligned} \tag{82a}$$

where

$$\begin{aligned}
m_x = &\int_{-h/2}^{h/2} (\bar{X}_1 + \Delta X_1)x_3\, dx_3 + \frac{h}{2}\Big[u_{1,1}(\sigma_{31}{}^+ + \sigma_{31}{}^-) -\\
&\frac{h}{2} w_{,11}(\sigma_{31}{}^+ - \sigma_{31}{}^-) + u_{1,2}(\sigma_{32}{}^+ + \sigma_{32}{}^-) -\\
&\frac{h}{2} w_{,12}(\sigma_{32}{}^+ - \sigma_{32}{}^-) - w_{,1}(\sigma_{33}{}^+ + \sigma_{33}{}^-) +\\
&\bar{\sigma}_{13}{}^+ + \bar{\sigma}_{13}{}^-\Big]
\end{aligned} \tag{82b}$$

$$\begin{aligned}
m_y = &\int_{-h/2}^{h/2} (\bar{X}_2 + \Delta X_2)x_3\, dx_3 + \frac{h}{2}\Big[u_{2,1}(\sigma_{31}{}^+ + \sigma_{31}{}^-) -\\
&\frac{h}{2} w_{,21}(\sigma_{31}{}^+ - \sigma_{31}{}^-) + u_{2,2}(\sigma_{32}{}^+ + \sigma_{32}{}^-) -\\
&\frac{h}{2} w_{,22}(\sigma_{32}{}^+ - \sigma_{32}{}^-) - w_{,2}(\sigma_{33}{}^+ + \sigma_{33}{}^-) +\\
&\bar{\sigma}_{23}{}^+ + \bar{\sigma}_{23}{}^-\Big]
\end{aligned} \tag{82c}$$

$$\begin{aligned}
q = &\int_{-h/2}^{h/2} (\bar{X}_3 + \Delta X_3)\, dx_3 + w_{,1}(\sigma_{31}{}^+ - \sigma_{31}{}^-) +\\
&w_{,2}(\sigma_{32}{}^+ - \sigma_{32}{}^-) + \bar{\sigma}_{33}{}^+ - \bar{\sigma}_{33}{}^-
\end{aligned} \tag{82d}$$

The force and moment boundary conditions can be found by changing Eqs. (77–79) into normal and tangential coordinates and substituting into Eqs. (49–53)

$$\begin{aligned}
\bar{F}_{nn} + \Delta F_{nn} &= N_n u_{n,n} - M_n w_{,nn} + N_{nt} u_{n,t} - M_{nt} w_{,nt} -\\
&\qquad Q_n w_{,n} + D(u_{n,n} + \nu u_{t,t})\\
\bar{F}_{nt} + \Delta F_{nt} &= N_n u_{t,n} - M_n w_{,tn} + N_{nt} u_{t,t} - M_{nt} w_{,tt} -\\
&\qquad Q_n w_{,t} + Gh(u_{n,t} + u_{t,n})
\end{aligned} \tag{83}$$

$$\begin{aligned}
\bar{F}_{n3} + \Delta F_{n3} &= N_n w_{,n} + N_{nt} w_{,t} + \bar{Q}_n\\
\bar{M}_{nn} + \Delta M_{nn} &= M_n u_{n,n} - M_n{}^* w_{,nn} + M_{nt} u_{n,t} - M_{nt}{}^* w_{,nt} -\\
&\qquad Q_n{}^* w_{,n} - \mathscr{D}(w_{,nn} + \nu w_{,tt})\\
\bar{M}_{nt} + \Delta M_{nt} &= M_n u_{t,n} - M^* w_{,nt} + M_{nt} u_{t,t} - M_{nt}{}^* w_{,tt} -\\
&\qquad Q_n{}^* w_{,t} - \mathscr{D}(1-\nu) w_{,nt}
\end{aligned} \tag{84}$$

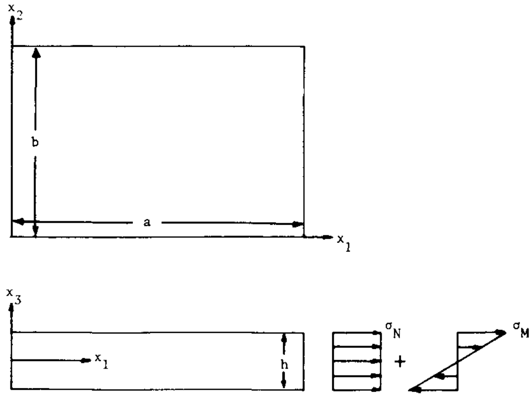

Fig. 1 Rectangular plate subjected to uniform tensile plus uniform bending stresses in the x_1-direction.

The effective shear (Kirchhoff boundary condition) is given by

$$(\bar{M}_{nt}+\Delta M_{nt})_{,t}+\bar{F}_{n3}+\Delta F_{n3} \equiv \bar{S}_{n3}+\Delta S_{n3}$$

where

$$\bar{S}_{n3}+\Delta S_{n3} = N_n w_{,n}+N_{nt} w_{,t}+[M_n u_{t,n}-M_n{}^* w_{,nt}+ M_{nt} u_{t,t}-M_{nt}{}^* w_{,tt}-Q_n{}^* w_{,t}-\mathscr{D}(1-\nu)w_{,nt}]_{,t}+\bar{Q}_n \tag{85}$$

$\bar{Q}_n$ can be determined by expressing either Eq. (43) or (44) in normal and tangential coordinates, and by using Eqs. (77) and (79). Alternative displacement boundary conditions are

$$\left.\begin{aligned} u_n &= u_{nn} \\ u_t &= u_{nt} \\ w &= w_n \end{aligned}\right\} \tag{86}$$

The same results can be obtained by proceeding as in Sec. IV only the displacement field and the stress–displacement relations are modified as in Eqs. (77) and (78).

VII. Example Problem

Consider a simply supported Mindlin plate in a state of initial stress. The state of initial stress is

$$\sigma_{11} = \sigma_N + 2x_3\sigma_M/h \tag{87}$$

with all other initial stresses assumed to be zero. σ_N and σ_M are taken to be constants so the initial stress field is uniform. It is comprised of a tension (or compression) plus bending, see Fig. 1. From Eqs. (38) the only nonzero initial stresses are

$$\left.\begin{aligned} N_x &= h\sigma_N \\ M_x &= h^2\sigma_M/6 \\ M_x{}^* &= h^3\sigma_N/12 \end{aligned}\right\} \tag{88}$$

Lateral loads and body forces are taken to be zero.

$$f_x, f_y, q, m_x, m_y = 0 \tag{89}$$

The equations of motion (40–44) simplify to

$$\left.\begin{aligned}
&M_x\psi_{1,11}+N_x u_{1,11}+D(u_{1,11}+\nu u_{2,21})+Gh(u_{1,22}+u_{2,12}) = \rho h\ddot{u}_1 \\
&M_x\psi_{2,11}+N_x u_{2,11}+Gh(u_{1,21}+u_{2,11})+D(u_{2,22}+\nu u_{1,12}) = \rho h\ddot{u}_2 \\
&N_x w_{,11}+\kappa^2 G^* h(\psi_{1,1}+w_{,11})+\kappa^2 G^* h(\psi_{2,2}+w_{,22}) = \rho h\ddot{w} \\
&M_x{}^*\psi_{1,11}+M_x u_{1,11}+\mathscr{D}(\psi_{1,11}+\nu\psi_{2,21}) + \frac{Gh^3}{12}(\psi_{1,22}+\psi_{2,12})-\kappa^2 G^* h(\psi_1+w_{,1}) = \frac{\rho h^3}{12}\ddot{\psi}_1 \\
&M_x{}^*\psi_{2,11}+M_x u_{2,11} + \frac{Gh^3}{12}(\psi_{1,21}+\psi_{2,11})+ \mathscr{D}(\psi_{2,22}+\nu\psi_{1,12})-\kappa^2 G^* h(\psi_2+w_{,2}) = \frac{\rho h^3}{12}\ddot{\psi}_2
\end{aligned}\right\} \tag{90}$$

The boundary condition Eqs. (49–54) are, for the simply supported plate, on the x_1 = constant edges

$$\left.\begin{aligned} w &= 0 \\ \psi_2 &= 0 \\ u_2 &= 0 \\ \bar{F}_{11}+\Delta F_{11} &= M_x\psi_{1,1}+N_x u_{1,1}+D(u_{1,1}+\nu u_{2,2}) = 0 \\ \bar{M}_{11}+\Delta M_{11} &= M_x{}^*\psi_{1,1}+M_x u_{1,1}+\mathscr{D}(\psi_{1,1}+\nu\psi_{2,2}) = 0 \end{aligned}\right\} \tag{91}$$

and on the x_2 = constant edges

$$\left.\begin{aligned} w &= 0 \\ \psi_1 &= 0 \\ u_1 &= 0 \\ \bar{F}_{22}+\Delta F_{22} &= M_y\psi_{2,2}+N_y u_{2,2}+D(u_{2,2}+\nu u_{1,1}) = 0 \\ \bar{M}_{22}+\Delta M_{22} &= M_y{}^*\psi_{2,2}+M_y u_{2,2}+\mathscr{D}(\psi_{2,2}+\nu\psi_{1,1}) = 0 \end{aligned}\right\} \tag{92}$$

Displacements of the following form satisfy the spatial part of Eqs. (90) and boundary conditions (91) and (92).

$$\left.\begin{aligned}
w &= W_{mn}\sin\left(\frac{m\pi x_1}{a}\right)\sin\left(\frac{n\pi x_2}{b}\right)\exp(i\omega t) \\
\psi_1 &= \Psi_{1mn}\cos\left(\frac{m\pi x_1}{a}\right)\sin\left(\frac{n\pi x_2}{b}\right)\exp(i\omega t) \\
\psi_2 &= \Psi_{2mn}\sin\left(\frac{m\pi x_1}{a}\right)\cos\left(\frac{n\pi x_2}{b}\right)\exp(i\omega t) \\
u_1 &= U_{1mn}\cos\left(\frac{m\pi x_1}{a}\right)\sin\left(\frac{n\pi x_2}{b}\right)\exp(i\omega t) \\
u_2 &= U_{2mn}\sin\left(\frac{m\pi x_1}{a}\right)\cos\left(\frac{n\pi x_2}{b}\right)\exp(i\omega t)
\end{aligned}\right\} \tag{93}$$

The displacements can be used to study the buckling and free vibration behavior of the simply supported rectangular plate under initial stress (87). Equations (93) are substituted into Eqs. (90). This leads to the following eigenvalue problem for the determination of buckling loads, natural frequencies, and coefficients.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{pmatrix} U_{1nm} \\ U_{2nm} \\ W_{nm} \\ \Psi_{1nm} \\ \Psi_{2nm} \end{pmatrix} = 0 \tag{94}$$

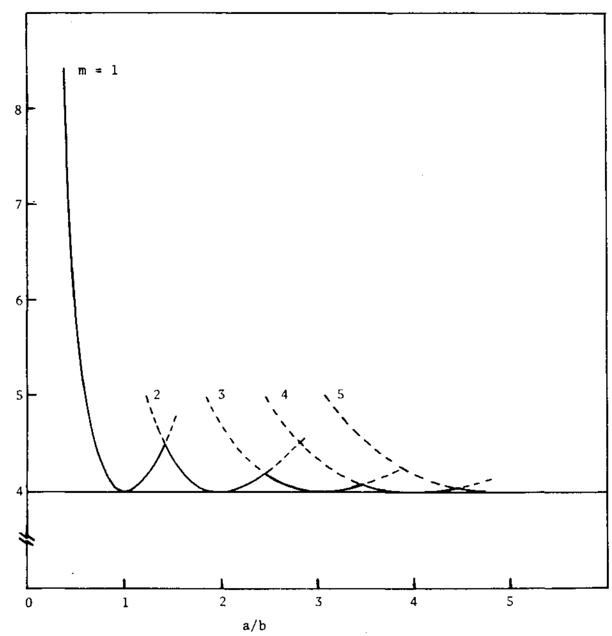

Fig. 2 Buckling curves for the simply supported rectangular plate when a/h = 100, β = 0, S = 0.0001, n = 1.

where

$$
\left.
\begin{aligned}
a_{11} &= \frac{m^2\pi^2}{a^2}(N_x+D) + \frac{Ghn^2\pi^2}{b^2} - \rho h\omega^2 \\
a_{12} &= \frac{mn\pi^2}{ab}(\nu D+Gh) \\
a_{13} &= 0 \\
a_{14} &= \left(\frac{m\pi}{a}\right)^2 M_x \\
a_{15} &= 0 \\
a_{21} &= \frac{mn\pi^2}{ab}(\nu D+Gh) \\
a_{22} &= \left(\frac{m\pi}{a}\right)^2 (N_x+Gh)+D\left(\frac{n\pi}{b}\right)^2 - \rho h\omega^2 \\
a_{23} &= 0 \\
a_{24} &= 0 \\
a_{25} &= \left(\frac{m\pi}{a}\right)^2 M_x \\
a_{31} &= 0 \\
a_{32} &= 0 \\
a_{33} &= \left(\frac{m\pi}{a}\right)^2 (N_x+\kappa^2G^*h) + \left(\frac{n\pi}{b}\right)^2 \kappa^2G^*h-\rho h\omega^2 \\
a_{34} &= \frac{m\pi}{a}\kappa^2G^*h \\
a_{35} &= \frac{n\pi}{b}\kappa^2G^*h \\
a_{41} &= \left(\frac{m\pi}{a}\right)^2 M_x \\
a_{42} &= 0 \\
a_{43} &= \frac{m\pi}{a}\kappa^2G^*h \\
a_{44} &= \left(\frac{m\pi}{a}\right)^2 (M_x{}^*+\mathscr{D}) + \frac{Gh^3}{12}\left(\frac{n\pi}{b}\right)^2 + \kappa^2G^*h - \frac{\rho h^3}{12}\omega^2 \\
a_{45} &= \frac{nm}{ab}\pi^2\left(\nu\mathscr{D} + \frac{Gh^3}{12}\right) \\
a_{51} &= 0 \\
a_{52} &= \left(\frac{m\pi}{a}\right)^2 M_x \\
a_{53} &= \frac{n\pi}{b}\kappa^2G^*h \\
a_{54} &= \frac{nm}{ab}\pi^2\left(\frac{Gh^3}{12} + \nu\mathscr{D}\right) \\
a_{55} &= \left(\frac{m\pi}{a}\right)^2\left(M_x{}^* + \frac{Gh^3}{12}\right) + \mathscr{D}\left(\frac{n\pi}{b}\right)^2 + \kappa^2G^*h - \frac{\rho h^3}{12}\omega^2
\end{aligned}
\right\} \quad (95)
$$

Equations (94) are now nondimensionalized. Factor a/π from the fourth and fifth columns which serves to replace Ψ_{1nm} and Ψ_{2nm} by $a\Psi_{1nm}/\pi$ and $a\Psi_{2nm}/\pi$, respectively. Multiply the first row by $a^2\mathscr{D}/D\pi^2$, multiply the second row by $(a/\pi D)^2$, multiply the third row by $a^2/(\pi^2\kappa^2G^*h)$, multiply the fourth row by $a^3/(\pi^3\mathscr{D})$ and multiply the fifth row by $a^3/(\pi^3\mathscr{D})$. Then, define the following parameters:

$\beta = \sigma_M/\sigma_N$

$Q = \kappa^2G^*h/\mathscr{D} = 12\kappa^2G^*/hD$

$M = \rho h^3/12\mathscr{D} = \rho h/D$

$S = Eh^2/G^*b^2(1-\nu^2) = Dh/b^2G^*$ transverse isotropy parameter

$S_x = N_x/\mathscr{D} = 12N_x/h^2D$

$k = (b/\pi)^2S_x = 12b^2N_x/\pi^2h^2D$ buckling coefficient

$K = (a/h)^2/\kappa^2$

$\Omega^2 = \rho a^4\omega^2/h\pi^2\kappa^2D$ nondimensional frequency

then taking $\kappa^2 = \pi^2/12$ gives[4]

$Q = \pi^2G^*/hD = S(\pi/b)^2$

$(3K)^{1/2} = 6a/\pi h$

$k/K = (b/a)^2N_x/D$

$\Omega^2/K = \rho ha^2\omega^2/\pi^2D$

$kS = N_x/\kappa^2G^*h = 12N_x/\pi^2G^*h$

Ω^2 has been nondimensionalized with respect to the first Bernoulli-Euler beam frequency. Because of the nondimensionalization procedure Eq. (94) is replaced by

$$
\begin{pmatrix}
b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\
b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\
b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\
b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\
b_{51} & b_{52} & b_{53} & b_5{}^4 & b_{55}
\end{pmatrix}
\begin{pmatrix}
U_{1nm} \\ U_{2nm} \\ W_{nm} \\ a\Psi_{1nm}/\pi \\ a\Psi_{2nm}/\pi
\end{pmatrix} = 0
$$

where

$b_{11} = m^2+n^2\left(\frac{1-\nu}{2}\right)(a/b)^2+m^2(a/b)^2k/K-\Omega^2/K$

$b_{12} = \left(\frac{1+\nu}{2}\right)mna/b$

$b_{13} = 0$

$b_{14} = m^2(a/b)^2(k/K)\beta/(3K)^{1/2}$

$b_{15} = 0$

$b_{21} = \left(\frac{1+\nu}{2}\right)mna/b$

$b_{22} = (na/b)^2 + \left(\frac{1-\nu}{2} + (a/b)^2k/K\right)m^2-\Omega^2/K$

$b_{23} = 0$

$b_{24} = 0$

$b_{25} = (ma/b)^2(k/K)\beta/(3K)^{1/2}$

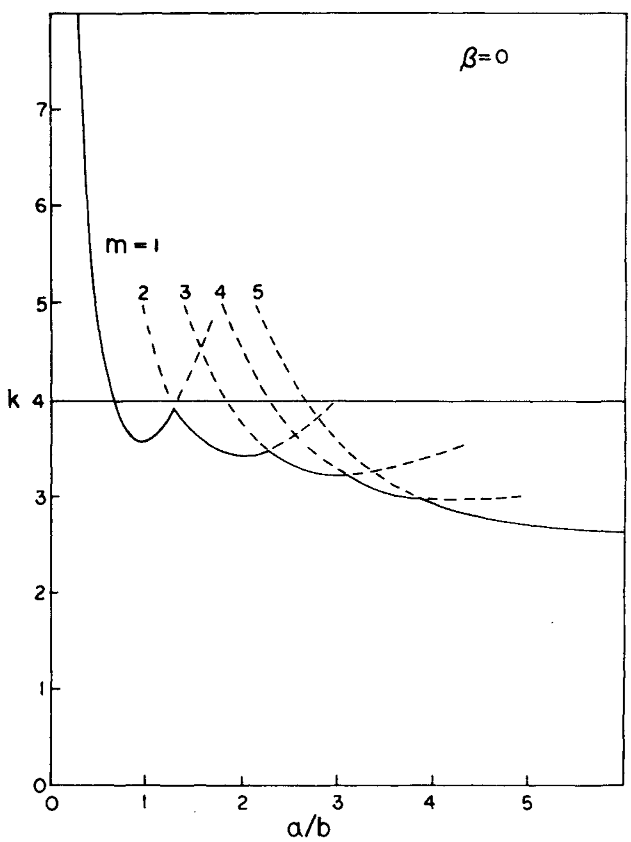

Fig. 3 Buckling curves for the simply supported rectangular plate when $a/h = 10$, $\beta = 0$, $S = 0.05$, $n = 1$.

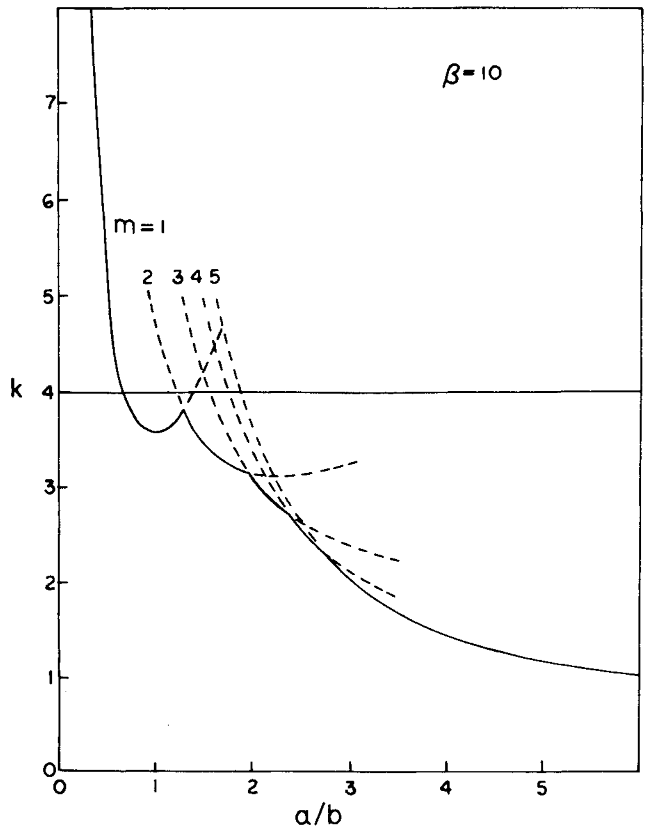

Fig. 4 Buckling curves for the simply supported rectangular plate when $a/h = 10$, $\beta = 10$, $S = 0.05$, $n = 1$.

$$b_{31} = 0$$
$$b_{32} = 0$$
$$b_{33} = (a/b)^2[m^2 + kSm^2 + (na/b)^2] - S\Omega^2$$
$$b_{34} = m(a/b)^2$$
$$b_{35} = n(a/b)^3$$
$$b_{41} = S\beta(a/b)^2 m^2 k/(3K)^{1/2}$$
$$b_{42} = 0$$
$$b_{43} = m(a/b)^2$$
$$b_{44} = S\left\{m^2[(a/b)^2 k/K + 1] + \left(\frac{1-\nu}{2}\right)(a/b)^2 n^2 + (a/b)^2/S - \Omega^2/K\right\}$$
$$b_{45} = Smn(a/b)(1+\nu)/2$$
$$b_{51} = 0$$
$$b_{52} = S\beta(a/b)^2 m^2 k/(3K)^{1/2}$$
$$b_{53} = n(a/b)^3$$
$$b_{54} = Smn(a/b)(1+\nu)/2$$
$$b_{55} = S\{m^2[(a/b)^2 k/K + (1-\nu)/2] + (na/b)^2 + (a/b)^2/S - \Omega^2/K\}$$

Several cases of buckling will be examined. Buckling is studied by letting $\Omega^2 = 0$ and seeking values of k (buckling coefficient) for various values of the other parameters. Note that negative k will correspond to compressive initial stress. Take $\nu = 0.3$.

In the first case just σ_N is acting so that $\sigma_M = 0$ and therefore $\beta = 0$. Further, the plate is assumed to be thin by taking $a/h = 100$ and $S = 10^{-4}$. The small value of S corresponds to a large value of G^* which implies that there is little transverse shearing. It is expected that the results will correspond to classical buckling theory of thin plates. In the usual manner a plot is made of k vs a/b for various values of m with $n = 1$ (actually the absolute value of k is plotted vs a/b since buckling will only occur in the instances considered here when the initial stress is compressive). The results are shown in Fig. 2. These results agree with classical plate buckling theory.[7] This was to be expected since Eqs. (40–44) reduce down to Eqs. (80–82), for the thin plate assumptions, and these further reduce down to the classical buckling equations in the case of the present problem.

In the second case a thick plate is considered. Again, with just σ_N acting $\beta = 0$. Taking $a/h = 10$ and $S = 0.05$ plots of k, vs a/b were made for $m = 1$–5 when $n = 1$. It can be seen in Fig. 3 that the buckling coefficient decreases with the inclusion of transverse shearing. How much it decreases depends on S and therefore on the degree of transverse isotropy.

The third and final case to be considered is buckling in the presence of an initial moment. Thus, σ_N and σ_M are acting. Taking $\beta = 10$, $a/h = 10$, and $S = 0.05$ the problem will be the same as the second case except that there is a uniform initial moment distribution. Plots were made of k vs a/b for $m = 1$–5 and $n = 1$. It can be seen in Fig. 4 that the initial moment distribution greatly reduces the buckling strength of the plate. These results must be looked at slightly askance from a quantitative point of view. This is because including an initial moment distribution could mean that initial displacement gradients are not negligible and therefore could not be dropped in going from Eqs. (13) and (14) to Eqs. (18) and (19). This, of course, means a new set of equations would have to be derived using the procedures given in this paper.

VIII. Conclusion

The preliminary results indicate that the transverse shear effects can decrease the stability of the plate and that initial bending stresses can seemingly cause a drastic reduction in stability. These results, though far from being complete, indicate some of the many interesting effects that can be studied with the present equations. Particularly interesting problems for future study can be visualized for nonconservative buckling problems as well as the effects of various initial stress fields on wave propagation and vibrations of thick plates.

Appendix

Referring to Fig. 5 and following accepted procedures,[1] the stress vector equation of equilibrium of the deformed body is given by

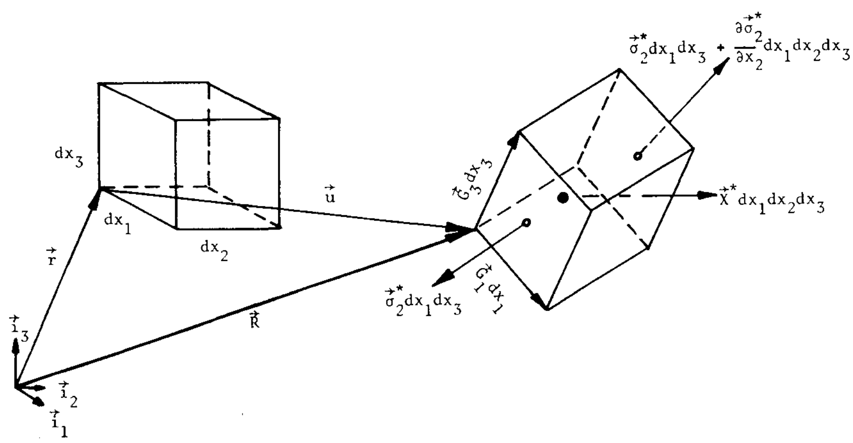

Fig. 5 Equilibrium of a deformed element.

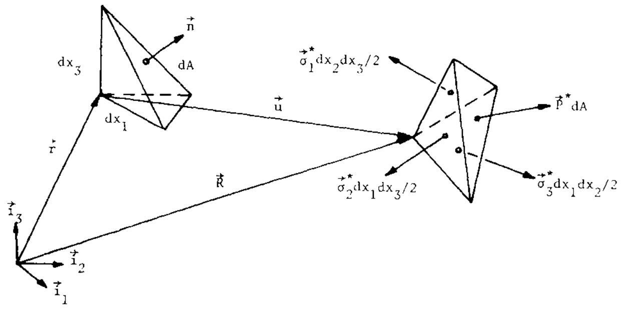

Fig. 6 Equilibrium with a surface traction.

$$\partial \boldsymbol{\sigma}_i^*/\partial x_i + \mathbf{X}^* = 0 \tag{A1}$$

where $\boldsymbol{\sigma}_i^*$ is the stress vector referred to the undeformed ith face area and $\mathbf{X}^*$ is the body force vector referred to the undeformed volume. Resolving $\boldsymbol{\sigma}_i^*$ in the nonorthogonal lattice vector ($\mathbf{G}_j$) directions yields

$$\boldsymbol{\sigma}_i^* = \sigma_{ij}\mathbf{G}_j \tag{A2}$$

where σ_{ij}^* are the Trefftz components of stress, referred to the undeformed ith face area, which can be shown to be symmetric.[1] Since $\mathbf{G}_j = (\partial \mathbf{R}/\partial x_j)$ where $\mathbf{R}$ is the final state position vector it is seen that§

$$\mathbf{G}_j = (\delta_{js} + \partial u_s/\partial x_j)\mathbf{i}_s \tag{A3}$$

where $\mathbf{i}_s$ is the unit vector in the sth orthogonal Cartesian direction. Decomposing $\mathbf{X}^*$ into its $\mathbf{i}_s$ components

$$\mathbf{X}^* = X_s^* \mathbf{i}_s \tag{A4}$$

§ Note that $\mathbf{R} = \mathbf{r} + \mathbf{u} = (X_s + u_s)\mathbf{i}_s$; therefore $(\partial \mathbf{R}/\partial x_j) = [\delta_{js} + (\partial u_s/\partial x_j)]\mathbf{i}_s$.

and putting the results of Eqs. (A2–A4) into Eq. (A1) yields the following scalar equations of equilibrium:

$$\frac{\partial}{\partial x_i}[(\delta_{js} + \partial u_s/\partial x_j)\sigma_{ij}^*] + X_s^* = 0 \tag{A5}$$

Referring to Fig. 6, $\mathbf{p}^*$ is the prescribed traction referred to the undeformed oblique face, and noting that $dA(\mathbf{i}_i \cdot \mathbf{n}) = dA_i/2$, the equilibrium of the deformed tetrahedron is given by

$$\mathbf{p}^* = (\mathbf{i}_i \cdot \mathbf{n})\boldsymbol{\sigma}_i^* \equiv n_i \boldsymbol{\sigma}_i^* \tag{A6}$$

and by using Eqs. (A2) and (A3) and defining the $\mathbf{i}_s$ components of $\mathbf{p}$ as p_s^*, (A6) becomes

$$p_s^* = \sigma_{ij}^* n_i [\delta_{js} + (\partial u_s/\partial x_j)] \tag{A7}$$

Equations (A5) and (A7) are the desired nonlinear equations that describe the equilibrium condition and the surface tractions, respectively.

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Buckling of Polar Orthotropic Annular Plates under Uniform Internal Pressure

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The problem of buckling of polar orthotropic annular plates under uniform internal pressure is analyzed by the classical Rayleigh-Ritz method. Direct application of the method with simple polynomials as admissible functions is found to be inconvenient for large hole sizes, particularly, in the case of both edges clamped. In such situations, it is shown that the method in conjunction with a coordinate transformation introduced is convenient. Detailed numerical investigations have been carried out with regard to the convergence of solutions as well as the behavior of rounding errors in computations. Accurate estimates of critical buckling loads are obtained for various hole sizes and rigidity ratios and for all combinations of clamped, simply supported, and free edge conditions. A comparison with the results of isotropic plates has brought out some interesting features.

Nomenclature

a, b = radii of inner and outer edges, respectively
c = $a^2/(b^2 - a^2)$
C, S, F = clamped, simply supported, and free edge conditions, respectively
D = $Eh^3/12(1-\nu^2)$
D_r = $E_r h^3/12(1-\nu_r\nu_\theta)$
D_θ = $E_\theta h^3/12(1-\nu_r\nu_\theta)$
D_1 = $\nu_r D_\theta = \nu_\theta D_r$
E_r, E_θ = Young's moduli in radial and tangential directions, respectively
h = thickness of plate
k = $(E_\theta/E_r)^{1/2} = (D_\theta/D_r)^{1/2}$
p_i = uniform in-plane radial pressure at the inner edge
r, θ = polar coordinates of a point in midplane of plate
T = potential energy due to in-plane forces during bending
u_m, v_m = admissible functions
V = strain energy due to bending
$W(r)$ = lateral displacement
y = $(r^2 - a^2)/(b^2 - a^2)$
ν_r, ν_θ = Poisson's ratios
σ_r, σ_θ = normal stresses in radial and tangential directions, respectively
$\sigma_{r\theta}$ = shear stress

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